

# Elliptic Stochastic Partial Differential Equations with Two Reflecting Walls

Wen Yue, Tusheng Zhang

Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, England, UK

## Abstract

In this article, we study elliptic stochastic partial differential equations with two reflecting walls  $h^1$  and  $h^2$ , driven by multiplicative noise. The existence and uniqueness of the solutions are established.

*Keywords:* elliptic stochastic partial differential equations; reflecting walls; elliptic deterministic obstacle problems; random measures.

**AMS Subject Classification:** Primary 60H15 Secondary 60F10, 60F05.

## 1 Introduction

In this paper we will consider the following elliptic stochastic partial differential equations (SPDEs) with Dirichlet boundary condition on a bounded domain  $D$  of  $\mathbb{R}^k$ ,  $k = 1, 2, 3$ .

$$-\Delta u(x) + f(x; u(x)) = \eta(x) - \xi(x) + \sigma(x; u(x))\dot{W}(x), \quad x \in D, \quad (1.1)$$

where  $\{\dot{W}(x), x \in D\}$  is a white noise in  $D$ . We are looking for a continuous random field  $u(x)$ ,  $x \in D$  which is the solution of equation (1.1) satisfying  $h^1(x) \leq u(x) \leq h^2(x)$ , where  $h^1$  and  $h^2$  are given two walls. When  $u(x)$  hits  $h^1(x)$  or  $h^2(x)$ , the additional forces are added to prevent  $u$  from leaving  $[h^1, h^2]$ . These forces are expressed by random measures  $\xi$  and  $\eta$  in equation (1.1) which play a similar role as the local time in the usual Skorokhod equation constructing Brownian motions with reflecting barriers. SPDEs with two reflecting walls can be used to model the evolution of random interfaces near two hard walls, see T. Funaki and S. Olla [4]. For nonlinear elliptic PDEs with measures as right side or boundary condition, we refer to Boccardo, Gallouet [1] and Rockner, Zegarlinski [8].

For elliptic SPDEs without reflection, R. Buckdahn and E. Pardoux in [2] established the existence and uniqueness of the solutions of nonlinear elliptic stochastic partial differential equations driven by additive noise. Based on this, elliptic SPDEs with reflection at zero driven by additive noise, have been studied by David Nualart and Samy Tindel in [6].

In our present paper, we will study the elliptic SPDEs with two reflecting walls driven by multiplicative noise. This is the first time to consider the case of multiplicative noise. We will establish the existence and uniqueness of the solutions. A similar problem for reflected stochastic heat equations has been studied by Nualart and Pardoux in [5], Donati-Martin and Pardoux in [3], Yang and Zhang in [12] and by Xu and Zhang in [11]. Our approaches were inspired by the ones in [5], [6], [7] and [11].

The rest of the paper is organized as follows. In Section 2, we lay down the framework of the paper. In Section 3, we study deterministic reflected elliptic PDEs and obtain some a priori estimates. The main result is established in Section 4.

## 2 Framework

Let  $D$  be an open bounded subset of  $\mathbb{R}^k$ , with  $k \in \{1, 2, 3\}$ . Consider a Gaussian family of random variables  $\{W = W(B), B \in \mathcal{B}(D)\}$ , where  $\mathcal{B}(D)$  is the Borel  $\sigma$ -field on  $D$ , defined in a complete probability space  $(\Omega, \mathcal{F}, P)$ , such that  $E(W(B)) = 0$  and

$$E(W(A)W(B)) = |A \cap B|, \quad (2.1)$$

where  $|A \cap B|$  denotes the Lebesgue measure of the set  $A \cap B$ . We want to study a reflected nonlinear stochastic elliptic equation with Dirichlet condition driven by multiplicative noise:

$$-\Delta u(x) + f(x, u(x)) = \sigma(x; u(x))\dot{W}(x), \quad (2.2)$$

where  $x \in D$  while  $h^1(x) \leq u(x) \leq h^2(x)$ ,  $\dot{W}(x)$  is the formal derivative of  $W$  with respect to the Lebesgue measure and the symbol  $\Delta$  denotes the Laplace operator on  $L^2(D)$ . If  $u(x)$  hits  $h^1(x)$  or  $h^2(x)$ , additional forces are added in order to prevent  $u$  from leaving  $[h^1, h^2]$ . Such an effect will be expressed by adding extra(unknown) terms  $\xi$  and  $\eta$  in (2.2) which play a similar role as the local time in the usual Skorokhod equation constructing Brownian motions with reflecting boundaries.

$\mathcal{C}_0^\infty(D)$  denotes the set of infinitely differentiable functions on  $D$  with compact supports. We will denote by  $(\cdot, \cdot)$  the scalar product in  $L^2(D)$ , and by  $\|\cdot\|_\infty$  the supremum norm on  $D$ . Let  $f, \sigma : D \times \mathbb{R} \rightarrow \mathbb{R}$  be measurable functions. We will also denote by  $f(u)$  the function  $f(u)(x) = f(x, u(x))$ ,  $\sigma(u)$  the function  $\sigma(u)(x) = \sigma(x, u(x))$ . We introduce the following hypotheses on  $f$  and  $\sigma$ :

(F1) The function  $f$  is locally bounded, continuous and nondecreasing as a function of the second variable.

( $\Sigma$  1) The function  $\sigma$  is Lipschitz continuous:

$$|\sigma(x, z_1) - \sigma(x, z_2)| \leq C_\sigma |z_1 - z_2|.$$

(H1) The walls  $h^i(x), i = 1, 2$ , are continuous functions satisfying  $h^1(x) < h^2(x)$  for  $x \in D$  and  $h^1(x) \leq 0 \leq h^2(x)$  for  $x \in \partial D$ .

The solution to Eq(1.1) will be a triplet  $(u, \eta, \xi)$  such that  $h^1(x) \leq u(x) \leq h^2(x)$  on  $D$  which satisfies Eq(1.1) in the sense of distributions, and  $\eta(dx), \xi(dx)$  are random measures on  $D$  which force the process  $u$  to be in the interval  $[h^1, h^2]$ . More precisely, a rigorous definition of the solution to Eq(1.1) is given as follows:

**Definition 2.1** A triplet  $(u, \eta, \xi)$  defined on a complete probability space  $(\Omega, \mathcal{F}, P)$  is a solution to the SPDE (1.1), denoted by  $(0; f; \sigma; h^1, h^2)$ , if

- (i)  $\{u(x), x \in D\}$  is a continuous random field on  $D$  satisfying  $h^1(x) \leq u(x) \leq h^2(x)$  and  $u|_{\partial D} = 0$  a.s.  
(ii)  $\eta(dx)$  and  $\xi(dx)$  are random measures on  $D$  such that  $\eta(K) < \infty$  and  $\xi(K) < \infty$  for all compact subset  $K \subset D$ .  
(iii) For all  $\phi \in \mathcal{C}_0^\infty(D)$ , we have

$$-(u, \Delta\phi) + (f(u), \phi) = \int_D \phi(x) \sigma(u) W(dx) + \int_D \phi(x) \eta(dx) - \int_D \phi(x) \xi(dx). \quad P - a.s. \quad (2.3)$$

$$(iv) \int_D (u(x) - h^1(x)) \eta(dx) = \int_D (h^2(x) - u(x)) \xi(dx) = 0.$$

### 3 Deterministic obstacle problem

Let  $h^1, h^2$  be as in Section 2, and  $f$  satisfies (F1). Let  $v(x) \in C(D)$  with  $v|_{\partial D} = 0$ . Consider a deterministic elliptic PDE with two reflecting walls:

$$\begin{cases} -\Delta z + f(z + v) = \eta - \xi \\ h^1 \leq z + v \leq h^2 \\ z|_{\partial D} = 0. \end{cases} \quad (3.1)$$

Here is a precise definition of the solution of equation (3.1).

**Definition 3.1** A triplet  $(z, \eta, \xi)$  is called a solution to the PDE (3.1) if

- (i)  $z = z(x); x \in D$  is a continuous function satisfying  $h^1(x) \leq z(x) + v(x) \leq h^2(x)$ ,  $z|_{\partial D} = 0$ .  
(ii)  $\eta(dx)$  and  $\xi(dx)$  are measures on  $D$  such that  $\eta(K) < \infty$  and  $\xi(K) < \infty$  for all compact subset  $K \subset D$ .  
(iii) For all  $\phi \in \mathcal{C}_0^\infty(D)$  we have

$$-(z, \Delta\phi) + (f(z + v), \phi) = \int_D \phi(x) \eta(dx) - \int_D \phi(x) \xi(dx). \quad (3.2)$$

$$(iv) \int_D (z(x) + v(x) - h^1(x)) \eta(dx) = \int_D (h^2(x) - z(x) - v(x)) \xi(dx) = 0.$$

The following result is the existence and uniqueness of the solutions of the PDE with two reflecting walls (3.1).

**Theorem 3.1** Equation (3.1) admits a unique solution  $(z, \eta, \xi)$ .

We first consider the problem of a single reflecting barrier, denoted by  $(0; f; h^1)$ :

$$\begin{cases} -\Delta z + f(z + v) = \eta(x) \\ z + v \geq h^1 \\ z|_{\partial D} = 0 \\ \int_D (z + v - h^1) \eta(dx) = 0, \end{cases} \quad (3.3)$$

where the coefficient  $f$  satisfies (F1) and  $h^1$  satisfies (H1) in Section 2.

In the next lemma, we give the existence and uniqueness of the solution of  $(0; f; h^1)$ , and it follows from Theorem 2.2 in David Nualart and Samy Tindle [6] using similar methods.

**Lemma 3.1** *Let  $v$  be a continuous function on  $\bar{D}$  such that  $v|_{\partial D} = 0$ . There exists a unique pair  $(z, \eta)$  such that:*

- (i)  $z$  is a continuous function on  $\bar{D}$  such that  $z|_{\partial D} = 0$  and  $z + v \geq h^1$ .
- (ii)  $\eta$  is a measure on  $D$  such that  $\eta(K) < \infty$  for any compact set  $K \subset D$ .
- (iii) For every  $\phi \in \mathcal{C}_k^\infty(\mathcal{D})$ , we have

$$-(z, \Delta\phi) + (f(z + v), \phi) = \int_D \phi(x) \eta(dx).$$

$$(iv) \int_D (z(x) + v(x) - h^1(x)) \eta(dx) = 0.$$

Theorem 2.2 from David Nualart and Samy Tindel:

Let  $v$  be a continuous function on  $\bar{D}$  such that  $v|_{\partial D} = 0$ . There exist a unique pair  $(z, \eta)$  such that:

- (i)  $z$  is a continuous function on  $\bar{D}$  such that  $z|_{\partial D} = 0$  and  $z \geq -v$ .
- (ii)  $\eta$  is a measure on  $D$  such that  $\eta(K) < \infty$  for any compact set  $K \subset D$ .
- (iii) For every  $\phi \in \mathcal{C}_k^\infty(\mathcal{D})$ , we have

$$-(z, \Delta\phi) + (f(z + v), \phi) = \int_D \phi(x) \eta(dx).$$

$$(iv) \int_D (z(x) + v(x)) \eta(dx) = 0.$$

Next lemma is a comparison theorem for the PDE with reflection.

**Lemma 3.2** *(comparison)*

Let  $(z_1, \eta_1)$  and  $(z_2, \eta_2)$  be solutions to single reflection problems  $(0; f_1, h_1)$  and  $(0; f_2, h_2)$  respectively as in (3.3). If  $f_1 \leq f_2$ , and  $h_1 \geq h_2$ , for every  $x \in D$ , then we have  $z_1(x) \geq z_2(x)$ .

PROOF. Let  $z_1^\epsilon$  and  $z_2^\epsilon$  be the solutions of the following PDEs:

$$\begin{cases} -\Delta z_1^\epsilon(x) + f_1(z_1^\epsilon + v)(x) = \frac{1}{\epsilon}(z_1^\epsilon + v - h_1)^-(x) \\ z_1^\epsilon|_{\partial D} = 0. \end{cases} \quad (3.4)$$

$$\begin{cases} -\Delta z_2^\epsilon(x) + f_2(z_2^\epsilon + v)(x) = \frac{1}{\epsilon}(z_2^\epsilon + v - h_2)^-(x) \\ z_2^\epsilon|_{\partial D} = 0. \end{cases} \quad (3.5)$$

According to [6],  $z_1^\epsilon \rightarrow z_1$  and  $z_2^\epsilon \rightarrow z_2$  uniformly on  $\bar{D}$  as  $\epsilon \rightarrow 0$ .  
Let  $\psi = z_2^\epsilon - z_1^\epsilon$ , then

$$\begin{cases} -\Delta\psi + f_2(z_2^\epsilon + v) - f_1(z_1^\epsilon + v) = \frac{1}{\epsilon}[(z_2^\epsilon + v - h_2)^- - (z_1^\epsilon + v - h_1)^-] \\ \psi|_{\partial D} = 0. \end{cases} \quad (3.6)$$

Multiplying (3.6) by  $\psi^+$ , we obtain

$$(-\Delta\psi, \psi^+) + (f_2(z_2^\epsilon + v) - f_1(z_1^\epsilon + v), \psi^+) = \frac{1}{\epsilon}([(z_2^\epsilon + v - h_2)^- - (z_1^\epsilon + v - h_1)^-], \psi^+) \quad (3.7)$$

Note that,

$$-(\Delta\psi, \psi^+) = (\nabla\psi, \nabla\psi^+) = (\nabla\psi^+, \nabla\psi^+) = \|\nabla\psi^+\|_{L^2(D)}^2 \geq 0. \quad (3.8)$$

If  $\psi^+(x) \neq 0$ , we have  $z_2^\epsilon(x) > z_1^\epsilon(x)$ . Because  $f_2$  is increasing and  $f_1 \leq f_2$ , we also have

$$(f_2(z_2^\epsilon + v) - f_1(z_1^\epsilon + v), \psi^+) \geq 0. \quad (3.9)$$

Since  $h_1 \geq h_2$ , we have  $z_2^\epsilon(x) + v(x) - h_2(x) \geq z_1^\epsilon(x) + v(x) - h_1(x)$  and then

$$\frac{1}{\epsilon}([(z_2^\epsilon(x) + v(x) - h_2(x))^- - (z_1^\epsilon(x) + v(x) - h_1(x))^-], \psi^+(x)) \leq 0. \quad (3.10)$$

Thus it follows from (3.7), (3.8), (3.9) and (3.10) that:

$$\|\nabla\psi^+\|_{L^2(D)}^2 = 0.$$

Hence, by the boundary condition  $\psi^+|_{\partial D} = 0$ , we get  $\psi^+ = 0$  and then  $z_2^\epsilon \leq z_1^\epsilon$ , for every  $\epsilon > 0$ . Hence, the lemma follows immediately by taking  $\epsilon \rightarrow 0$ .  $\square$

**Lemma 3.3** *Let  $v$  and  $\hat{v}$  be given continuous functions and let  $z^{\epsilon, \delta}$  be a unique solution to the following deterministic PDE:*

$$\begin{cases} -\Delta z^{\epsilon, \delta}(x) + f(z^{\epsilon, \delta} + v)(x) = \frac{1}{\delta}(z^{\epsilon, \delta}(x) + v(x) - h^1(x))^- - \frac{1}{\epsilon}(z^{\epsilon, \delta}(x) + v(x) - h^2(x))^+, \\ z^{\epsilon, \delta}|_{\partial D} = 0. \end{cases} \quad (3.11)$$

We also denote by  $\hat{z}^{\epsilon, \delta}$  the solution to the above PDE replacing  $v$  by  $\hat{v}$ . Then we have,  $\|z^{\epsilon, \delta} - \hat{z}^{\epsilon, \delta}\|_\infty \leq \|v - \hat{v}\|_\infty$ , where  $\|w\|_\infty = \sup_{x \in D} |w(x)|$ .

**PROOF.** Define  $w(x) = z^{\epsilon, \delta}(x) - \hat{z}^{\epsilon, \delta}(x) - l$ , where  $l = \|v - \hat{v}\|_\infty$ . Then,  $w$  satisfies the following PDE:

$$\begin{aligned} -\Delta w + f(z^{\epsilon, \delta} + v) - f(\hat{z}^{\epsilon, \delta} + \hat{v}) &= \frac{1}{\delta}[(z^{\epsilon, \delta} + v - h^1)^- - (\hat{z}^{\epsilon, \delta} + \hat{v} - h^1)^-] \\ &\quad - \frac{1}{\epsilon}[(z^{\epsilon, \delta} + v - h^2)^+ - (\hat{z}^{\epsilon, \delta} + \hat{v} - h^2)^+] \end{aligned} \quad (3.12)$$

Set

$$F_{\epsilon,\delta}(u) = f(u) - \frac{1}{\delta}(u - h^1)^- + \frac{1}{\epsilon}(u - h^2)^+$$

Now we note that, if  $w^+(x) > 0$ , we have  $z^{\epsilon,\delta}(x) + v(x) > \hat{z}^{\epsilon,\delta}(x) + \hat{v}(x)$  and hence

$$\begin{cases} f(z^{\epsilon,\delta} + v)(x) \geq f(\hat{z}^{\epsilon,\delta} + \hat{v})(x) \\ \frac{1}{\delta}[(z^{\epsilon,\delta} + v - h^1)^-(x) - (\hat{z}^{\epsilon,\delta} + \hat{v} - h^1)^-(x)] \leq 0 \\ \frac{1}{\epsilon}[(z^{\epsilon,\delta} + v - h^2)^+(x) - (\hat{z}^{\epsilon,\delta} + \hat{v} - h^2)^+(x)] \geq 0, \end{cases} \quad (3.13)$$

Consequently, on the set  $\{x \in D; w^+(x) > 0\}$ , we have

$$F_{\epsilon,\delta}(z^{\epsilon,\delta} + v)(x) - F_{\epsilon,\delta}(\hat{z}^{\epsilon,\delta} + \hat{v})(x) \geq 0 \quad (3.14)$$

On the other hand, multiplying (3.12) by  $w^+$ , we obtain:

$$-(\Delta w, w^+) + (F_{\epsilon,\delta}(z^{\epsilon,\delta} + v) - F_{\epsilon,\delta}(\hat{z}^{\epsilon,\delta} + \hat{v}), w^+) = 0$$

Because

$$-(\Delta w, w^+) = \|\nabla w^+\|_{L^2(D)}^2 \geq 0,$$

it follows from (3.15) that

$$\|\nabla w^+\|_{L^2(D)}^2 = 0$$

and

$$(F_{\epsilon,\delta}(z^{\epsilon,\delta} + v) - F_{\epsilon,\delta}(\hat{z}^{\epsilon,\delta} + \hat{v}), w^+) = 0$$

Taking into account the fact  $w^+ = 0$  on  $\partial D$ , we deduce  $w^+ = 0$ . Hence  $z^{\epsilon,\delta} - \hat{z}^{\epsilon,\delta} \leq l$ . Interchanging the role of  $z^{\epsilon,\delta}$  and  $\hat{z}^{\epsilon,\delta}$ , we prove the lemma.  $\square$

The next lemma is a straight consequence of the above lemma.

**Lemma 3.4** *Let  $v$  and  $\hat{v}$  be given continuous functions and let  $(z^\epsilon, \eta^\epsilon)$  and  $(\hat{z}^\epsilon, \hat{\eta}^\epsilon)$  be the unique solutions to single reflection problems  $(0; f + \frac{(\cdot + v - h^2)^+}{\epsilon}; h^1)$  and  $(0; f + \frac{(\cdot + \hat{v} - h^2)^+}{\epsilon}; h^1)$ , respectively. Then we have  $\|z^\epsilon - \hat{z}^\epsilon\|_\infty \leq \|v - \hat{v}\|_\infty$ .*

Proof of Theorem 3.1:

Denote by  $z^\epsilon$  the solution of the following single barrier problem:

$$\begin{cases} -\Delta z^\epsilon + f(z^\epsilon + v) + \frac{1}{\epsilon}(z^\epsilon + v - h^2)^+ = \eta^\epsilon \\ z^\epsilon + v \geq h^1 \\ \int_D (z^\epsilon + v - h^1) \eta^\epsilon(dx) = 0, \end{cases} \quad (3.15)$$

By the construction in [6], it is known that  $\eta^\epsilon(dx) = \lim_{\delta \rightarrow 0} \frac{(z^{\epsilon,\delta} + v - h^1)^-}{\delta}(dx)$  and it means that the measure  $\frac{(z^{\epsilon,\delta} + v - h^1)^-}{\delta}(dx)$  converges to  $\eta^\epsilon(dx)$  in the sense of distribution on  $D$ . According

to lemma 3.2(comparison):  $z^\epsilon(x)$  is decreasing as  $\epsilon \downarrow 0$ . Since  $z^\epsilon(x) \geq h^1(x) - v(x)$ ,  $z^\epsilon(x)$  converge to some function  $z(x)$  as  $\epsilon \rightarrow 0$ . Using similar arguments as in the proof of Lemma 3.2 in [6], we can show that the function  $z(x)$  is also continuous.

Next we prove that  $z(x)$  is a solution of the reflected PDE with two reflected walls

$$\begin{cases} -\Delta z + f(z + v) = \eta - \xi \\ h^1 \leq z + v \leq h^2 \\ \int_D (z + v - h^1) \eta(dx) = \int_D (h^2 - z - v) \xi(dx) = 0. \end{cases} \quad (3.16)$$

Step1:

Now for  $\psi \in C_0^\infty(D)$ ,  $z^\epsilon$  satisfies the following integral equation:

$$-(\Delta z^\epsilon, \psi) + (f(z^\epsilon + v), \psi) + \left(\frac{1}{\epsilon}(z^\epsilon + v - h^2)^+, \psi\right) = \int \psi(x) \eta^\epsilon(dx) \quad (3.17)$$

i.e.

$$-(z^\epsilon, \Delta \psi) + (f(z^\epsilon + v), \psi) = \int \psi(x) (\eta^\epsilon - \xi^\epsilon)(dx), \quad (3.18)$$

where  $\xi^\epsilon = \frac{(z^\epsilon + v - h^2)^+}{\epsilon}$ . The limit of the left hand side of (3.18) exists as  $\epsilon \rightarrow 0$ . Therefore  $\lim_{\epsilon \rightarrow 0} (\eta^\epsilon - \xi^\epsilon)$  exists in the space of distributions, i.e.

$$-(z, \Delta \psi) + (f(z + v), \psi) = \lim_{\epsilon \rightarrow 0} (\eta^\epsilon - \xi^\epsilon, \psi) \quad (3.19)$$

Next we want to show that both  $\lim_{\epsilon \rightarrow 0} \eta^\epsilon$  and  $\lim_{\epsilon \rightarrow 0} \xi^\epsilon$  exist. By Dini theorem, we know that  $z^\epsilon(x) \rightarrow z(x)$  uniformly on compact subsets of  $D$ . For  $\phi(x) \in C_0^\infty(D)$ , denote by  $K = \text{supp}(\phi)$ , the compact support of  $\phi$ . As  $h^1(x) < h^2(x)$  in  $D$ , there exists  $\theta_K > 0$  such that  $h^2(x) - h^1(x) \geq \theta_K$  on  $K$ . On the other hand, there exists  $\epsilon_0 > 0$ , such that for  $\epsilon < \epsilon_0$ ,  $|z^\epsilon(x) - z(x)| < \frac{\theta_K}{4}$  on  $K$ . Let  $\theta_K$  be chosen as above. Since

$$\text{supp} \eta^\epsilon \subseteq \{x : z^\epsilon(x) + v(x) = h^1(x)\},$$

and

$$\text{supp} \xi^\epsilon = \{x : z^\epsilon(x) + v(x) \geq h^2(x)\},$$

we have for  $\epsilon \leq \epsilon_0$ ,

$$\text{supp} \eta^\epsilon \cap K \subseteq \{x : z(x) - \frac{\theta_K}{4} + v(x) \leq h^1(x)\} \cap K := A_K,$$

and

$$\text{supp} \xi^\epsilon \cap K \subseteq \{x : z(x) + \frac{\theta_K}{4} + v(x) \geq h^2(x)\} \cap K := B_K,$$

for  $\epsilon < \epsilon_0$ .

By the choice of  $\theta_K$ , we see that  $A_K \cap B_K = \emptyset$ . Thus, we can find  $\tilde{\phi}(x) \in C_0^\infty(D)$  such that  $\tilde{\phi} = \phi$  on  $A_K$ ,  $\text{supp} \tilde{\phi} \cap B_K = \emptyset$  and  $\text{supp} \tilde{\phi} \cap \text{supp} \xi^\epsilon = \emptyset$  for  $\epsilon < \epsilon_0$ . Hence,  $\lim_{\epsilon \rightarrow 0} (\eta^\epsilon, \phi) = \lim_{\epsilon \rightarrow 0} (\eta^\epsilon, \tilde{\phi}) = \lim_{\epsilon \rightarrow 0} (\eta^\epsilon - \xi^\epsilon, \tilde{\phi})$  exists. Therefore  $\eta^\epsilon \rightarrow \eta$  in the space of distributions.

Similarly,  $\xi^\epsilon \rightarrow \xi$ . Let  $\epsilon \rightarrow 0$  in equation (3.19) to see that  $(z, \eta, \xi)$  satisfies the following equation:

$$-(\Delta z, \psi) + (f(z + v), \psi) = \int_D \psi(x)(\eta - \xi)(dx). \quad (3.20)$$

Step 2:

Multiplying (3.17) by  $\epsilon$  and letting  $\epsilon \rightarrow 0$ , we get

$$0 = ((z + v - h^2)^+, \psi).$$

This implies  $z + v - h^2 \leq 0$ , i.e.  $z + v \leq h^2$ . Since  $h^1 \leq z^\epsilon + v$ , we see that  $h^1 \leq z + v$ . So  $h^1 \leq z + v \leq h^2$ .

Step 3:

Now let us show that

$$\int_D (z + v - h^1)\eta(dx) = 0$$

and

$$\int_D (z + v - h^2)\xi(dx) = 0.$$

By the definition of  $\xi^\epsilon = \frac{(z^\epsilon + v - h^2)^+}{\epsilon}$ ,  $\int_D (z^\epsilon + v - h^2)\xi^\epsilon(dx) \geq 0$ , and the uniform convergence of  $z^\epsilon$  on compact subsets, letting  $\epsilon \rightarrow 0$ , we have  $\int_D (z + v - h^2)\xi(dx) \geq 0$ . Hence we must have  $\int_D (z + v - h^2)\xi(dx) = 0$ . From the single reflecting barrier problem  $(0; -\frac{(\cdot + v - h^2)^+}{\epsilon}; h^1)$ , we know  $\int_D (z^\epsilon + v - h^1)\eta^\epsilon(dx) = 0$ . Then letting  $\epsilon \downarrow 0$ , we get  $\int_D (z + v - h^1)\eta(dx) = 0$ .

Step 4:

For any compact set  $K \subset D$ , since

$$-(\Delta z, \psi) + (f(z + v), \psi) = \int_D \psi(x)\eta(dx) - \int_D \psi(x)\xi(dx).$$

Choose a non-negative function  $\psi \in C_0^\infty(D)$  such that  $\psi(x) = 1$  on  $\text{supp}(\eta) \cap K$  and  $\psi(x) = 0$  on  $\text{supp}(\xi) \cap K$ ,

$$-(\Delta z, \psi) + (f(z + v), \psi) = \int_K \eta(dx) - 0,$$

So we get  $\eta(K) < \infty$ . Similarly,  $\xi(K) < \infty$ .

Uniqueness: Let  $(z, \eta, \xi)$  and  $(\bar{z}, \bar{\eta}, \bar{\xi})$  be solutions to a double reflection problem  $(0; f; h^1, h^2)$ . We set  $\Psi = z - \bar{z}$ . For any  $\psi \in C_0^\infty(D)$ , we have

$$\begin{aligned} & - \int_D \Psi(x) \Delta \psi(x) dx + \int_D [f(z + v) - f(\bar{z} + v)] \psi(x) dx \\ &= \int_D \psi(x) \eta(dx) - \int_D \psi(x) \xi(dx) - \int_D \psi(x) \bar{\eta}(dx) + \int_D \psi(x) \bar{\xi}(dx). \end{aligned} \quad (3.21)$$

From here, following the same arguments as that in the proof of Theorem 2.2 in [6], we can show that  $z = \bar{z}$ .



Recall that

$$\begin{aligned} \text{supp}\eta, \text{supp}\bar{\eta} &\subset \{x \in D : z + v = h^1\} =: A, \\ \text{supp}\xi, \text{supp}\bar{\xi} &\subset \{x \in D : z + v = h^2\} =: B. \end{aligned}$$

Because  $A \cap B = \emptyset$ , for any  $\psi \in C_0^\infty(D)$  with  $\text{supp}\psi \subset \text{supp}\eta \cup \text{supp}\bar{\eta}$ , it holds that  $\text{supp}\psi \cap \text{supp}\xi = \emptyset$  and  $\text{supp}\psi \cap \text{supp}\bar{\xi} = \emptyset$ . Applying equation (3.21) to such a function  $\psi$ , we deduce that  $\eta = \bar{\eta}$ . Similarly  $\xi = \bar{\xi}$ . Then the uniqueness is proved.  $\square$

## 4 Reflected SPDEs

Recall

$$\begin{cases} -\Delta u(x) + f(x, u(x)) = \sigma(x, u(x))\dot{W}(x) + \eta(x) - \xi(x) \\ u|_{\partial D} = 0 \\ h^1(x) \leq u(x) \leq h^2(x) \\ \int_D (u(x) - h^1(x))\eta(dx) = \int_D (h^2(x) - u(x))\xi(dx) = 0. \end{cases} \quad (4.1)$$

Let  $G_D(x, y)$  be the Green function on  $D$  associated to the Laplacian operator with Dirichlet boundary conditions. We recall from [2] (or [9]) that if  $k = 2$  or  $3$ ,

$$G_D(x, y) = G(x, y) - E_x(G(B_\tau, y)), x, y \in D$$

with

$$\begin{aligned} G(x, y) &= \frac{1}{2\pi} \log|x - y|, \quad \text{if } k = 2; \\ G(x, y) &= -\frac{1}{4\pi} |x - y|^{-1}, \quad \text{if } k = 3; \end{aligned}$$

and  $B_\tau$  is the random variable obtained by stopping a  $k$ -dimensional Brownian motion starting at  $x$  at its first exit time of  $D$ . For  $k = 1$ , if  $D = (0, 1)$ , then  $G_D(x, y) = (x \wedge y) - xy$ .

The main result of this paper is the following theorem.

**Theorem 4.1** *Assume that (F1), (H1) and ( $\Sigma 1$ ) with  $C_\sigma$  satisfying  $\exists p > 1$ ,*

$$[2^{2p-1}ac_p Br_D^{\lambda p-k} + 2^{2p-1}c_p(C_D)^{\frac{p}{2}}]C_\sigma^p < 1, \quad (4.2)$$

where  $c_p$  and  $a$  are universal constants appeared in the Burkholder's inequality, Komogorov's inequality,  $r_D$  is the diameter of the domain  $D$  (see (4.20), (4.21)),  $C_D = \sup_x \int_D |G_D(x, y)|^2 dy$ . And  $B$  is the constant appeared in the estimate of the Green function  $G_D$  in (4.13). And  $\lambda$  is any number in  $(0, 1]$  when the dimension  $k = 1$ ;  $\lambda$  is any number in  $(0, 1)$  when the dimension  $k = 2$ ;  $\lambda$  is any number in  $(0, \frac{1}{2})$  when the dimension  $k = 3$ .

Then there exists a unique solution  $(u, \eta, \xi)$  to the reflected SPDE Eq(1.1). Moreover,  $E(\|u\|_\infty)^p < \infty$ .

PROOF.

Existence:

We will use successive iteration:

Let

$$v_1(x) = \int_D G_D(x, y) \sigma(y; 0) W(dy). \quad (4.3)$$

As in [2], it is seen that  $v_1(x)$  is the solution of the following SPDE:

$$\begin{cases} -\Delta v_1(x) = \sigma(x; 0) \dot{W}(x) \\ v_1|_{\partial D} = 0 \end{cases} \quad (4.4)$$

and  $v_1(x) \in C(\bar{D})$ .

Denote by  $(z_1, \eta_1, \xi_1)$  be the unique random solution of the following reflected PDE:

$$\begin{cases} -\Delta z_1(x) + f(z_1 + v_1) = \eta_1(x) - \xi_1(x) \\ z_1|_{\partial D} = 0 \\ h^1(x) \leq z_1(x) + v_1(x) \leq h^2(x) \\ \int_D (z_1(x) + v_1(x) - h^1(x)) \eta_1(dx) = \int_D (h^2(x) - z_1(x) - v_1(x)) \xi_1(dx) = 0. \end{cases} \quad (4.5)$$

Set  $u_1 = z_1 + v_1$ . Then we can easily verify that  $(u_1, \eta_1, \xi_1)$  is the unique solution of the following reflected SPDE:

$$\begin{cases} -\Delta u_1(x) + f(x; u_1) = \sigma(x; 0) \dot{W}(x) + \eta_1(x) - \xi_1(x) \\ u_1|_{\partial D} = 0 \\ h^1(x) \leq u_1(x) \leq h^2(x) \\ \int_D (u_1(x) - h^1(x)) \eta_1(dx) = \int_D (h^2(x) - u_1(x)) \xi_1(dx) = 0. \end{cases} \quad (4.6)$$

Iterating this procedure, suppose  $u_{n-1}$  has been defined. Let

$$v_n(x) = \int_D G_D(x, y) \sigma(y; u_{n-1}) W(dy), \quad (4.7)$$

and  $(z_n, \eta_n, \xi_n)$  be the unique random solution of the following reflected PDE:

$$\begin{cases} -\Delta z_n(x) + f(z_n + v_n) = \eta_n(x) - \xi_n(x) \\ z_n|_{\partial D} = 0 \\ h^1(x) \leq z_n(x) + v_n(x) \leq h^2(x) \\ \int_D (z_n(x) + v_n(x) - h^1(x)) \eta_n(dx) = \int_D (h^2(x) - z_n(x) - v_n(x)) \xi_n(dx) = 0. \end{cases} \quad (4.8)$$

Set  $u_n = z_n + v_n$ . Then  $(u_n, \eta_n, \xi_n)$  is the unique solution of the following reflected SPDE:

$$\begin{cases} -\Delta u_n(x) + f(x; u_n(x)) = \sigma(x; u_{n-1}(x))\dot{W}(x) + \eta_n - \xi_n \\ u_n|_{\partial D} = 0 \\ h^1(x) \leq u_n(x) \leq h^2(x) \\ \int_D (u_n(x) - h^1(x))\eta_n(dx) = \int_D (h^2(x) - u_n(x))\xi_n(dx) = 0. \end{cases} \quad (4.9)$$

From the proof of Lemma 3.4 (also Lemma 3.1 in [6]), we have

$$\|z_n - z_{n-1}\|_\infty \leq \|v_n - v_{n-1}\|_\infty, \quad (4.10)$$

hence

$$\|u_n - u_{n-1}\|_\infty \leq 2\|v_n - v_{n-1}\|_\infty. \quad (4.11)$$

Namely,

$$\begin{aligned} & (\|u_n - u_{n-1}\|_\infty) \\ & \leq 2 \sup_{x \in D} \left| \int_D G_D(x, y) (\sigma(y; u_{n-1}) - \sigma(y; u_{n-2})) W(dy) \right|. \end{aligned} \quad (4.12)$$

Set

$$I(x) = \int_D G_D(x, y) (\sigma(y; u_{n-1}) - \sigma(y; u_{n-2})) W(dy).$$

Then  $\forall p \geq 1$ ,

$$\begin{aligned} & E[|I(x) - I(y)|^p] \\ & = E \left| \int_D (G_D(x, z) - G_D(y, z)) (\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})) W(dz) \right|^p \\ & \leq c_p E \left[ \int_D |G_D(x, z) - G_D(y, z)|^2 \cdot |\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})|^2 dz \right]^{\frac{p}{2}} \\ & \leq c_p E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p \left[ \int_D |G_D(x, z) - G_D(y, z)|^2 dz \right]^{\frac{p}{2}}, \end{aligned}$$

where  $c_p$  is a Burkholder constant only related to  $p$ .

Similarly as the proof of Theorem 3.3 in [9], we have

$$\|G_D(x, z) - G_D(y, z)\|_{L^2(D)}^2 \leq B|x - y|^{2\lambda}, \quad (4.13)$$

where  $\lambda = 1$  when  $k = 1$ ,  $\lambda$  is arbitrarily close to 1 when  $k = 2$ , and  $\lambda$  is arbitrarily close to  $\frac{1}{2}$  when  $k = 3$ . Then,

$$E|I(x) - I(y)|^p \leq E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p c_p B |x - y|^{\lambda p}. \quad (4.14)$$

We next show that  $u_n$  converges uniformly on  $D$ . Let  $K \subset D$  be any compact subset of  $D$ .  $\forall x, y \in K$ , from Kolmogorov lemma (Lemma 3.1 in [3]), we deduce that for  $\forall p > \frac{k}{\lambda}$ ,

$$|I(x) - I(y)|^p \leq (N(w))^p |x - y|^{\lambda p - k} (\log(\frac{\gamma}{|x - y|}))^2, \quad (4.15)$$

$$E(N^p) \leq ac_p B E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p, \quad (4.16)$$

where  $a$  is a universal constant independent of  $K$ . Choosing  $y = x_0 \in K$ , we see that

$$\begin{aligned} E[\sup_{x \in K} |I(x)|^p] &\leq 2^{p-1} E[\sup_{x \in K} |I(x) - I(x_0)|^p] + 2^{p-1} E|I(x_0)|^p \\ &\leq 2^{p-1} E(N^p) r_D^{\lambda p - k} + 2^{p-1} E|I(x_0)|^p \\ &\leq 2^{p-1} ac_p B r_D^{\lambda p - k} E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p \\ &\quad + 2^{p-1} E|I(x_0)|^p, \end{aligned} \quad (4.17)$$

where  $r_D = \sup_{x, y \in D} |x - y|$  is the diameter of  $D$ . Furthermore,

$$\begin{aligned} E|I(x_0)|^p &= E \left| \int_D G_D(x_0, y) (\sigma(y; u_{n-1}) - \sigma(y; u_{n-2})) W(dy) \right|^p \\ &\leq c_p E \left[ \int_D |G_D(x_0, y)|^2 \cdot |\sigma(y; u_{n-1}) - \sigma(y; u_{n-2})|^2 dy \right]^{\frac{p}{2}} \\ &\leq c_p (C_D)^{\frac{p}{2}} E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p, \end{aligned} \quad (4.18)$$

where  $C_D = \sup_x \int_D |G_D(x, y)|^2 dy < \infty$ . So we have

$$\begin{aligned} &E[\sup_{x \in K} |I(x)|^p] \\ &\leq 2^{p-1} ac_p B r_D^{\lambda p - k} E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p \\ &\quad + 2^{p-1} c_p (C_D)^{\frac{p}{2}} E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p. \end{aligned} \quad (4.19)$$

Since the constants on the right side of (4.19) are independent of the compact subset  $K$ , by Fatou's Lemma we deduce that

$$\begin{aligned} &E[\sup_{x \in D} |I(x)|^p] \\ &\leq 2^{p-1} ac_p B r_D^{\lambda p - k} E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p \\ &\quad + 2^{p-1} c_p (C_D)^{\frac{p}{2}} E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p. \end{aligned} \quad (4.20)$$

Now it follows from (4.12) and (4.20) that

$$\begin{aligned} &E(\|u_n - u_{n-1}\|_\infty)^p \\ &\leq [2^{2p-1} ac_p B r_D^{\lambda p - k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] \\ &\quad \times E \|\sigma(z; u_{n-1}) - \sigma(z; u_{n-2})\|_\infty^p \\ &\leq [2^{2p-1} ac_p B r_D^{\lambda p - k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] C_\sigma^p \end{aligned}$$

$$\begin{aligned}
& \times E \|u_{n-1} - u_{n-2}\|_\infty^p \\
& \leq \dots \\
& \leq \left( [2^{2p-1} a c_p B r_D^{\lambda p-k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] C_\sigma^p \right)^{n-1} E \|u_1 - u_0\|_\infty^p
\end{aligned} \tag{4.21}$$

Since

$$[2^{2p-1} a c_p B r_D^{\lambda p-k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] C_\sigma^p < 1,$$

we obtain from (4.21) that for any  $m \geq n \geq 1$ ,

$$E(\|u_m - u_n\|_\infty)^p \rightarrow 0,$$

as  $n, m \rightarrow \infty$ .

Hence, there exists a continuous random field  $u(\cdot) \in C(D)$ , such that

$$E(\|u\|_\infty^p) < \infty, \tag{4.22}$$

and

$$\lim_{n \rightarrow \infty} E(\|u_n - u\|_\infty)^p = 0. \tag{4.23}$$

Next we will show that  $u$  is a solution of Eq(4.1).

Set

$$v(x) = \int_D G_D(x, y) \sigma(y; u) W(dy). \tag{4.24}$$

As the proof of (4.20), we have

$$\lim_{n \rightarrow \infty} E \|v_n - v\|_\infty^p = \lim_{n \rightarrow \infty} E \|u_{n-1} - u\|_\infty^p = 0. \tag{4.25}$$

From the inequality (4.10), there exists a continuous random field  $z(x)$  on  $D$  such that  $\lim_{n \rightarrow \infty} E \|z_n - z\|_\infty^p = 0$ . So  $z_n$  converges to  $z$  uniformly on  $D$ . Similar to the proof of Theorem 3.1, we can show that  $\eta(dx) = \lim_{n \rightarrow \infty} \eta_n(dx)$ ,  $\xi(dx) = \lim_{n \rightarrow \infty} \xi_n(dx)$  exist almost surely and  $(z, \eta, \xi)$  is the solution of equation (3.1) with the above given  $v$ . Put  $u(x) = z(x) + v(x)$ . It is easy to verify  $(u, \eta, \xi)$  is a solution to the SPDE(4.1) with two reflecting walls.

Uniqueness:

Let  $(u_1, \eta_1, \xi_1)$  and  $(u_2, \eta_2, \xi_2)$  be two solutions of Eq(4.1). Set

$$v_1(x) = \int_D G_D(x, y) \sigma(y; u_1) W(dy), \tag{4.26}$$

$$v_2(x) = \int_D G_D(x, y) \sigma(y; u_2) W(dy), \tag{4.27}$$

and  $z_1 = u_1 - v_1$  and  $z_2 = u_2 - v_2$ . Then  $z_1, z_2$  are solutions of the following reflected random PDEs:

$$\left\{ \begin{array}{l} -\Delta z_1(x) + f(z_1 + v_1) = \eta_1(x) - \xi_1(x) \\ z_1|_{\partial D} = 0 \\ h^1(x) \leq z_1(x) + v_1(x) \leq h^2(x) \\ \int_D (z_1(x) + v_1(x) - h^1(x))\eta_1(dx) = \int_D (h^2(x) - z_1(x) - v_1(x))\xi_1(dx) = 0, \end{array} \right. \quad (4.28)$$

$$\left\{ \begin{array}{l} -\Delta z_2(x) + f(z_2 + v_2) = \eta_2(x) - \xi_2(x) \\ z_2|_{\partial D} = 0 \\ h^1(x) \leq z_2(x) + v_2(x) \leq h^2(x) \\ \int_D (z_2(x) + v_2(x) - h^1(x))\eta_2(dx) = \int_D (h^2(x) - z_2(x) - v_2(x))\xi_2(dx) = 0, \end{array} \right. \quad (4.29)$$

Similar to the inequality (4.10), we have

$$\|z_1 - z_2\|_\infty \leq \|v_1 - v_2\|_\infty. \quad (4.30)$$

Hence,

$$\begin{aligned} \|u_1 - u_2\|_\infty^p &\leq 2^p \|v_1 - v_2\|_\infty^p \\ &\leq 2^p \left( \sup_{x \in D} \left| \int_D G_D(x, y) (\sigma(y; u_1) - \sigma(y; u_2)) W(dy) \right|^p \right) \end{aligned} \quad (4.31)$$

As the proof of (4.21), we deduce from (4.31) that

$$\begin{aligned} E \|u_1 - u_2\|_\infty^p &\leq [2^{2p-1} a c_p C(p) r_D^{\lambda p - k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] C_\sigma^p \\ &\quad \times E \|u_1 - u_2\|_\infty^p. \end{aligned} \quad (4.32)$$

As

$$[2^{2p-1} a c_p C(p) r_D^{\lambda p - k} + 2^{2p-1} c_p (C_D)^{\frac{p}{2}}] C_\sigma^p < 1,$$

it follows that

$$u_1 = u_2 \quad a.s. \quad (4.33)$$

On the other hand, for  $\phi \in C_0^\infty(D)$ ,

$$\begin{aligned} &-(u_1(x) - u_2(x), \Delta \phi(x)) + (f(x, u_1(x)) - f(x, u_2(x)), \phi(x)) \\ &= \int_D [\sigma(x; u_1(x)) - \sigma(x; u_2(x))] \phi(x) W(dx) \\ &\quad + \int_D \phi(x) (\eta_1(dx) - \eta_2(dx)) - \int_D \phi(x) (\xi_1(dx) - \xi_2(dx)). \end{aligned} \quad (4.34)$$

Therefore we have

$$\int_D \phi(x)(\eta_1(dx) - \eta_2(dx)) - \int_D \phi(x)(\xi_1(dx) - \xi_2(dx)) = 0. \quad (4.35)$$

Recall that

$$\begin{aligned} \text{supp}\eta_1, \text{supp}\eta_2 &\subset \{x \in D : u_1(x) = h^1(x)\} =: A \\ \text{supp}\xi_1, \text{supp}\xi_2 &\subset \{x \in D : u_1(x) = h^2(x)\} =: B. \end{aligned}$$

Because  $A \cap B = \emptyset$ , for any  $\phi \in C_0^\infty(D)$  with  $\text{supp}\phi \subset (\text{supp}\eta_1 \cup \text{supp}\eta_2)$ , it holds that  $\text{supp}\phi \cap \text{supp}\xi_1 = \emptyset$  and  $\text{supp}\phi \cap \text{supp}\xi_2 = \emptyset$ . Applying equation(4.35) to such a function  $\phi$ , we deduce that  $\eta_1 = \eta_2$ . Similarly,  $\xi_1 = \xi_2$ . Then the uniqueness is proved.  $\square$

## References

- [1] L. Boccardo, T. Gallouet, Nonlinear elliptic and parabolic equations involving measure data, Journal of Functional Analysis. 87, 149-169, 1989.
- [2] R. Buckdahn, E. Pardoux, Monotonicity Methods for White Noise Driven Quasi-Linear SPDEs, Diffusion Processes and related problems in Analysis, Vol. I, (1990)219-233.
- [3] C. Donati-Martin, E. Pardoux, White noise driven SPDEs with reflection, Probability Theory and Related Fields 95, (1993) 1-24.
- [4] T. Funaki, S. Olla, Fluctuations for  $\nabla\phi$  interface model on a wall, Stochastic Process. Appl. 94(1)(2001)1-27.
- [5] D. Nualart, E. Pardoux, White Noise Driven Quasilinear SPDEs with Reflection, Probability Theory and Related Fields 93, (1992) 77-89.
- [6] D. Nualart, S. Tindel, Quasilinear Stochastic Elliptic Equations with Reflection, Stochastic Processes and their Applications 57, (1995) 73-82.
- [7] Y. Otobe, Stochastic Partial Differential Equations with Two Reflecting Walls, J.Math.Sci.Univ.Tokyo 13, (2006) 129-144.
- [8] M. Rockner, B. Zegarlinski, The Dirichlet problem for quasi-linear partial differential operators with boundary data given by a distribution, Stochastic Processes and their Applications, Volume 61, 1990, pp 301-326.

- [9] M. Sanz-Sole, I. Torrecilla, A fractional Poisson equation: existence, regularity and approximations of the solution, *Stochastics and Dynamics*, Vol. 09, No. 04,(2009) 519-548.
- [10] J. Walsh, An introduction to stochastic partial differential equations, in : P.L. Hennequin(Ed.), *Ecole d'ete de Probabilites St. Flour XIV*, in: *Lect. Notes Math.*, vol 1180, Springer, Berlin, Heidelberg, New York, 1986.
- [11] T. Xu, T. Zhang, White noise driven SPDEs with reflection: Existence, uniqueness and large deviation principles. *Stochastic Processes and their Applications* 119, (2009) 3453-3470.
- [12] J. Yang, T. Zhang, White noise driven SPDEs with two reflecting walls. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 14, (2011) 647-659.